

Susceptibility at the edge points of magnetization plateau of one-dimensional electron and spin systems

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We study the behavior of magnetization curve as a function of magnetic field in the immediate vicinity of the magnetization plateaus of one-dimensional (1D) electron systems within the bosonization formalism. First we discuss the plateau that is formed at the saturation magnetization of 1D electron system. We treat interactions between electrons in the lowest order of perturbation. We show that for isolated systems, where total number of electrons is not allowed to vary, magnetic susceptibility stays always finite away of half filling. Similar statement holds for many other magnetization plateaus supporting nonmagnetic gapless excitations encountered in 1D electron/spin systems in the absence of special symmetries or features responsible for the mode decoupling. We demonstrate it on example of the plateaus at irrational values of magnetization in doped modulated Hubbard chains. Finally we discuss the connection between the weak coupling description of saturation magnetization plateau and strong coupling description of zero magnetization plateau of attractively interacting electrons.

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I. INTRODUCTION

Magnetization process of one-dimensional (1D) electron systems is fairly well understood theoretically. For purely spin systems (where charge fluctuations are frozen) the magnetization curve (plotted magnetization versus magnetic field) can show plateau like behavior (at zero temperature) for certain rational values of magnetization¹ where magnetic excitations develop gap.²

Experimentally different materials have been synthesized exhibiting plateaus in their magnetization curve and having magnetic structure believed to be modeled by quasi-one-dimensional geometries. For example, dimerized spin- $S=1$ chain³ exhibits magnetization plateau at $1/2$ of saturation magnetization as predicted by theory.² Other quasi-1D compounds showing plateau at half of the saturation magnetization are spin- $S=1/2$ compounds.^{4,5}

Phase transitions at the edges of the magnetization plateau are usually described within the commensurate-incommensurate universality class.^{6,7} As a hallmark feature of commensurate-incommensurate phase transition the magnetization shows square root dependence on magnetic field in the vicinity to the edges of the plateau, giving rise to infinite magnetic susceptibility at the edge points. The simplest examples include magnetization plateau of spin- S chain at the saturation magnetization and a plateau of integer spin- S antiferromagnetic Heisenberg chain (or even leg spin- S ladder) at the zero magnetization and we depict them on Fig. 1.

When doped with charge carriers the plateaus can also appear at irrational (doping controlled) values of magnetization.⁸ Interesting property of such plateaus is that they can support nonmagnetic gapless excitations. Other examples of plateaus with similar feature may be found in frustrated spin chains, such as trimerized $S=1/2$ zigzag chain at $1/3$ of saturation magnetization where gapless chiral excitations coexist with gapped magnetic excitations⁹ or spin- S generalization of the integrable t - J chain doped with $S=1/2$ carriers.¹⁰

In this work we investigate in detail the edge points of plateaus with gapless nonmagnetic excitation where modes describing magnetic (gapped at the plateau) and nonmagnetic (gapless) excitations do not decouple. We will argue that in such situations, in drastic contrast to the square root behavior characterizing the commensurate-incommensurate phase transitions, magnetization depends linearly on magnetic field at the edges of plateau, thus rendering magnetic susceptibility finite at the edge points.

The paper is organized in the following way. First we describe within the effective theory bosonization approach^{11,12} the simplest plateau of saturation magnetization encountered in free electron system. Next we add interactions between electrons and treat them in the first order of coupling constants. We compare our results to exact results available for the Hubbard model for both positive and negative Hubbard couplings. In the case of attractive Hubbard interaction we corrected the missprint in the exact expression of the saturation susceptibility^{13,14} and convinced that bosonization recovers exact expression of saturation magnetization in the lowest order of onsite Hubbard interaction for

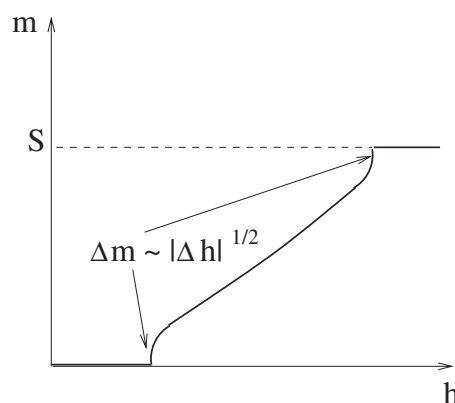


FIG. 1. Magnetization curve of integer spin S Heisenberg chain.

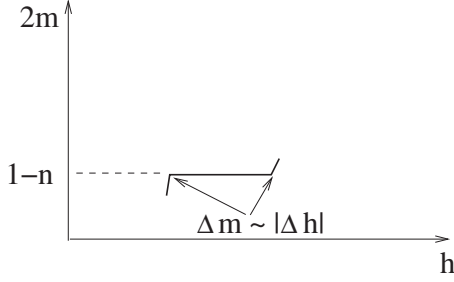


FIG. 2. Portion of the magnetization curve of a doped dimerized Hubbard chain.

both signs. We apply the similar theory to other magnetization plateaus that are encountered in more complicated problems, supporting gapless excitations, and in particular to the plateau in dimerized doped Hubbard chain at doping $\delta=1-n$, where n is a filling. As was shown in Ref. 8 at magnetization $m=\delta/2$ plateau is formed and we show that magnetization changes linearly with the field once plateau is closed by magnetic field on both sides as depicted on Fig. 2. Finally we discuss the connection between the weak coupling description of saturation magnetization plateau and strong coupling description of zero magnetization plateau of attractively interacting electrons.

In Appendixes A and C we give our bosonization rules and for completeness give exact expressions for saturation susceptibility for Hubbard model for both values of Hubbard coupling constant.

II. MODEL AND EFFECTIVE FIELD THEORY

The microscopic model, general enough for our purposes, is described by the following lattice Hamiltonian of the one-dimensional electrons:

$$H = - \sum_{i,j=1}^N \sum_{\sigma} t_i (c_{i,\sigma}^{\dagger} c_{i+1,\sigma} + c_{i+1,\sigma}^{\dagger} c_{i,\sigma}) + U \sum_{i=1}^N n_{i\uparrow} n_{i\downarrow} + V \sum_{i=1}^N \sum_{\sigma} n_{i\sigma} n_{i+1\sigma} - \frac{h}{2} \sum_{i=1}^N (n_{i\uparrow} - n_{i\downarrow}). \quad (1)$$

We will always work at the fixed filling. Above $n_{i,\sigma} = c_{i,\sigma}^{\dagger} c_{i,\sigma}$ is the number operator of spin $\sigma = \{\uparrow, \downarrow\}$ electrons at the i th site, U is the onsite coupling constant, V describes interaction between the same spin electrons on neighboring sites, and h is an external magnetic field. $t_i = t + t'(-1)^i$ is modulated hopping amplitude with period 2 (dimerized).

We will study the continuum limit of the above model using bosonization approach.^{11,12} Bosonization rules are summarized in Appendix A.

In bosonic formulation effective field theory of free fermions [setting $t' = U = V = 0$ in Eq. (1)] is just a direct sum of two decoupled Gaussian models, one for each spin component,

$$\mathcal{H} = \sum_{\sigma} \frac{v_{\sigma}}{2} \int dx \{ (\partial_x \phi_{\sigma})^2 + (\partial_x \theta_{\sigma})^2 \}, \quad (2)$$

where velocities are determined by the fillings: $v_{\sigma} = 2ta \sin \pi n_{\sigma}$, with a standing for lattice constant that we will set to 1.

III. SPIN SUSCEPTIBILITY AT THE SATURATION MAGNETIZATION

Magnetic susceptibility of the Hubbard model with fixed filling is always finite at the saturation magnetization for any interaction strength¹³⁻¹⁵ (including $U > 0$, $U < 0$, and $U = 0$) except for half filling¹⁶ and empty or completely filled bands. It is instructive first to develop bosonization analysis and reproduce the above result within the effective field theory language. This will serve to us as a warm up procedure for more complicated cases considered later. In this section we set $t' = 0$ and $t = 1$ in Eq. (1).

A. Bosonization for free electrons at saturation magnetization

First we set $U = V = 0$ in Eq. (1) and apply bosonization for free electrons at saturation magnetization. At saturation magnetization band of down-spin electrons gets empty, while the band of up-spin electrons gets either partially filled, if filling is less than half or completely filled exactly at half filling. To mimic the bottom of the down-spin electron band we add an appropriate relevant sine-Gordon potential to free bosonic theory describing the down-spin electrons (see Appendix B),

$$H = \frac{v_{\downarrow}}{2} [(\partial_t \phi_{\downarrow})^2 + (\partial_x \phi_{\downarrow})^2] + \frac{v_{\uparrow}}{2} [(\partial_t \phi_{\uparrow})^2 + (\partial_x \phi_{\uparrow})^2] + \frac{1}{2} (h_w + \Delta h) \frac{\partial_x \phi_{\downarrow}}{\sqrt{\pi}} + W \cos \sqrt{4\pi} \phi_{\downarrow} - \frac{\Delta h}{2} \frac{\partial_x \phi_{\uparrow}}{\sqrt{\pi}} - \Delta \mu \left(\frac{\partial_x \phi_{\downarrow}}{\sqrt{\pi}} + \frac{\partial_x \phi_{\uparrow}}{\sqrt{\pi}} \right). \quad (3)$$

Above $v_{\uparrow} = 2 \sin(\pi n)$ is the velocity of up-spin band at saturation, h_w closes the gap opened by the cosine term, and Δh is a deviation of magnetic field from the saturation value. Down-spin electron parameters v_{\downarrow} and W are selected in such a way to mimic optimally the dispersion at the bottom of the down-spin electron band. As it will become obvious we will not need their explicit values for calculating the magnetic susceptibility at the saturation magnetization. The chemical potential is a function of magnetic field (Lagrange multiplier enforcing constraint) and keeps the number of electrons fixed.

In bosonization the constraint of fixed particle number looks as

$$\langle \partial_x \phi_{\downarrow} + \partial_x \phi_{\uparrow} \rangle = 0. \quad (4)$$

On the other hand uniform magnetization density is given by the following expression:

$$\Delta m = \frac{\langle \partial_x \phi_{\uparrow} - \partial_x \phi_{\downarrow} \rangle}{2\sqrt{\pi}}. \quad (5)$$

At the saturation magnetization we have from Eq. (3)

$$\frac{\langle \partial_x \phi_\uparrow \rangle}{\sqrt{\pi}} = \frac{\Delta h/2 + \Delta\mu(h)}{\pi v_\uparrow} = \frac{\Delta h/2 + \Delta\mu(h)}{2\pi \sin \pi n}. \quad (6)$$

Using Eqs. (4) and (5) from Eq. (6) we can express the Lagrange multiplier $\Delta\mu$ as follows:

$$\Delta\mu = \pi v_\uparrow \Delta m - \Delta h/2. \quad (7)$$

In addition using the sine-Gordon Hamiltonian for down-spin electrons we have second equation^{6,7} (for details see Appendix B),

$$\begin{aligned} -\Delta n_\downarrow = \Delta m &= \frac{1}{\pi} \sqrt{|\Delta h/2 - \Delta\mu|} \\ &= \frac{1}{\pi} \sqrt{|\Delta h/2 - (\pi v_\uparrow \Delta m - \Delta h/2)|}, \end{aligned} \quad (8)$$

where for the last equality we have used Eq. (7). The correct root of Eq. (8), one vanishing with Δh , is

$$\Delta m = \frac{\Delta h}{\pi v_\uparrow} - \frac{(\Delta h)^2}{\pi v_\uparrow^3} + O(\Delta h^3). \quad (9)$$

Yielding for the magnetic susceptibility at the saturation magnetization,

$$\chi = \left. \frac{\partial m}{\partial h} \right|_{h_{sat}} = \frac{1}{\pi v_\uparrow} = \frac{1}{2\pi \sin \pi n}. \quad (10)$$

B. Including interactions perturbatively

From Eq. (10) we see that the susceptibility is finite except at half filling. Important observation is that this result is robust to interactions, one can treat interactions between electrons *perturbatively*, because the saturation magnetization plateau is caused by the band effect and exists already at the free electron level. To demonstrate this we will include perturbatively Hubbard coupling U and next-nearest-neighbor interaction V and obtain exact expression of susceptibility in the first order of these coupling constants. Interactions U and V in bosonization formulation close to saturation magnetization are translated as

$$\frac{U}{\pi} \partial_x \phi_\downarrow \partial_x \phi_\uparrow + \frac{V}{\pi} \partial_x \phi_\downarrow \partial_x \phi_\downarrow + \frac{V}{\pi} \partial_x \phi_\uparrow \partial_x \phi_\uparrow. \quad (11)$$

Other terms are either incommensurate, meaning that they rapidly oscillate, cancel out in the continuum limit or irrelevant (like umklapp process at half filling for the attractive case), and thus are neglected.

Using constraint (4) from Eq. (5) one can express magnetization by $\partial_x \phi_\uparrow$ simply by minimizing the energy,

$$\left\langle \frac{\delta H}{\delta \partial_x \phi_\uparrow} \right\rangle = 0, \quad (12)$$

where H is given by the sum of Eqs. (3) and (11). This fixes magnetization in the ground state configuration as follows:

$$\Delta m = \frac{\langle \partial_x \phi_\uparrow \rangle}{\sqrt{\pi}} = \frac{\Delta h/2 + \Delta\mu - (U-2V)\langle \partial_x \phi_\downarrow \rangle / \sqrt{\pi}}{\pi v_\uparrow}, \quad (13)$$

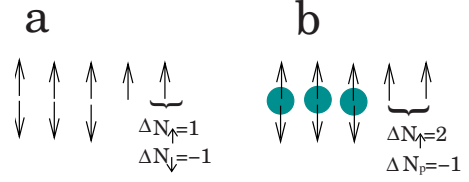


FIG. 3. (Color online) Variation in up-spin electron number in two interpretations: (a) up- and down-spin particles and (b) up-spin particles and pairs. Both parts correspond to the same configuration of spins, but due to different interpretations up-spin electron variations differ by factor of 2.

$$\Delta m \left(1 - \frac{U-2V}{\pi v_\uparrow} \right) = \frac{\Delta h/2 + \Delta\mu}{\pi v_\uparrow}, \quad (14)$$

$$\Delta\mu = \pi v_\uparrow \Delta m \left(1 - \frac{U-2V}{\pi v_\uparrow} \right) - \Delta h/2. \quad (15)$$

Now we need another equation to exclude $\Delta\mu$. We adopt a mean field decomposition of interaction terms and for the second equation we obtain

$$\begin{aligned} \Delta m &= \frac{1}{\pi} \sqrt{|\Delta h/2 - \Delta\mu + (U-2V)\Delta m|} \\ &= \frac{1}{\pi} \sqrt{|\Delta h - \Delta m[\pi v_\uparrow - 2(U-2V)]|}, \end{aligned} \quad (16)$$

$$\Delta m = \frac{\Delta h}{\pi \tilde{v}_\uparrow} + O(\Delta h^2), \quad (17)$$

where

$$\tilde{v}_\uparrow = v_\uparrow \left(1 - \frac{2(U-2V)}{\pi v_\uparrow} \right). \quad (18)$$

For the susceptibility at the saturation magnetization in the first order of coupling constants we obtain

$$\chi^{(n)} = \frac{1}{2\pi \sin(\pi n)} \left(1 + \frac{U-2V}{\pi \sin \pi n} \right). \quad (19)$$

The plus and the minus signs in front of U and V , respectively, are intuitively clear: repulsive Hubbard interaction facilitates to fully polarize the system while repulsion between the similar spin components hinders it. Moreover number of nearest neighbors along the chain is twice the number of sites, hence contribution from V is multiplied by factor of 2 in comparison with U .

In the case of the pure Hubbard model ($V=0$) we corrected a misprint of factor of 2 in Refs. 13 and 14 [one can repeat Bethe ansatz calculation close to saturation magnetization and get convinced that bosonization indeed gives correct result in the first order of U for both signs (see Appendix C)].

The above effective theory developed at the edge of the saturation plateau is not only valid in this particular case but also applies to other plateaus, e.g., those that are characterized by gapless nonmagnetic excitations which we will discuss in Sec. IV.

IV. PLATEAU OF DOPED DIMERIZED HUBBARD CHAIN AT IRRATIONAL VALUE OF MAGNETIZATION

In this section we will apply our bosonization description that we developed in previous sections to calculate magnetic susceptibility at the edges of plateau of doped dimerized Hubbard chain ($V=0, t' \neq 0$) at doping dependent value of magnetization $2m = \delta$.⁸ This plateau shares the similar physics of the saturation magnetization one. The exact equivalence with the saturation plateau is due to the fact that description is perturbative in interaction parameter, plateau exists already at the level of quadratic in Fermi fields Hamiltonian (due to the modulation of exchanges), and the limit $U \rightarrow 0$ is nonsingular.

The bosonized Hamiltonian at the plateau of dimerized doped Hubbard chain, (first setting Hubbard interaction to zero) is given by Eq. (3) since at this plateau one degree of freedom (up-spin) remains gapless, whereas down-spin electrons meet the commensurability condition and thus are gapped. Now we can add perturbatively Hubbard interaction or more general interactions and follow the steps that we developed in Sec. III B. In this way we will obtain susceptibilities in the lowest order in interaction constants. Situation is manifestly similar to the saturation magnetization case. From this equivalence we can answer a question⁸ on the critical properties at the edges of these plateaus. Namely, magnetization will increase always linearly with the magnetic field unless the total number of electrons is allowed to vary. If no constraint on the total number of particle is imposed, plateau itself will disappear.⁸ Once again, due to the linearly dispersive degree of freedom, potential Van Hove singularity from the bottom of the gapped mode is suppressed by the constraint of keeping total number of electrons fixed.

We think that similar effective theory can apply to the doping dependent magnetization plateau in an integrable spin- S generalization of the t - J chain doped with spin- $S=-1/2$ carriers.¹⁰ As explained within the Bethe ansatz method¹⁰ in this model quantum numbers of excitations carrying spin and charge get coupled at the right edge of the doping dependent plateau and this can cause the linear increase in magnetization as opposed to the naively anticipated square root behavior.

V. SPIN SUSCEPTIBILITY AT THE ONSET OF MAGNETIZATION

More subtle situation arises when magnetization plateau is formed due to the interactions (four fermion involving terms), and to describe correctly the edge of the plateau one needs to use nonperturbative approach. One example of such situation is the weak coupling limit of Hubbard model of attractively interacting electrons where the effects of curvature of the Fermi surface on commensurate-incommensurate transition in spin sector away of half filling were emphasized.¹⁷ However in the dilute limit of the same model a simple approach was developed¹⁷ using the reasoning along the lines suggested in Ref. 14 which allows us to obtain exact expression for susceptibility at the onset of magnetization.

Attractively interacting electrons

In this section we remind the bosonization description of zero magnetization plateau of attractively interacting two component fermions.¹⁷ In the dilute limit the similar approach as that at the saturation magnetization was developed, treating bound pairs and up-spin electrons as independent particles, but constrained by the total number of particles being fixed. The independence could be justified by a dilute limit, and treating bound pairs as hard core bosons can be justified by the inherent strong coupling nature of the dilute limit. Thus in the dilute limit the heuristic model developed in strong coupling¹⁷ is expected to yield exact expression for susceptibility. For completeness we remind the calculations developed in Ref. 17 in dilute limit. The Hamiltonian is given by the following expression:

$$H = \frac{v_{\uparrow}}{2} [(\partial_t \phi_{\uparrow})^2 + (\partial_x \phi_{\uparrow})^2] + \frac{v_p}{2} [(\partial_t \phi_p)^2 + (\partial_x \phi_p)^2] - \frac{(h_w + \Delta h)}{2} \frac{\partial_x \phi_{\uparrow}}{\sqrt{\pi}} + W \cos \sqrt{4\pi} \phi_{\uparrow} - \mu \left(\frac{\partial_x \phi_{\uparrow}}{\sqrt{\pi}} + \frac{2\partial_x \phi_p}{\sqrt{\pi}} + n \right), \quad (20)$$

where index p stands for the bound pairs. The above Hamiltonian is manifestly similar to the one describing the effective theory at the saturation plateau except that magnetic field does not couple to the pairs. In the same way as outlined before one can easily work out the magnetic susceptibility. The dispersion of uncompensated up-spin electrons at the critical field looks

$$E_{\uparrow}(k) = \sqrt{v_{\uparrow}^2 k^2 + \Delta^2} - \Delta \simeq \frac{v_{\uparrow}^2 k^2}{2\Delta}, \quad (21)$$

where Δ is a spin gap in the dilute limit and v_{\uparrow} is the velocity of uncompensated spins in the strong coupling (for Hubbard model, however their values do not appear in the expression of susceptibility). On the other hand bound pairs disperse linearly with velocity $v_p \neq 0$. Repeating analogous to the saturation case calculations we obtain for magnetization

$$\Delta m = \frac{1}{2\pi} \sqrt{\frac{\Delta}{v_{\uparrow}^2} (h - h_{cr} - v_p \pi \Delta m)} \quad (22)$$

and for susceptibility

$$\chi|_{n=\text{const}} = \frac{1}{\pi v_p}. \quad (23)$$

For Hubbard model $v_p = \frac{2\pi m}{|U|}$ is the charge (soft) mode velocity, so we get $\chi|_{n=\text{const}} = |U|/2\pi^2 n$. This is the low density limit of the exact strong coupling result: $\chi|_{n=\text{const}} = |U|/2\pi^2 n(1-n)^2$.¹⁴

Note in the picture of up-spin and bound pair particles $\Delta m = \frac{1}{2} \Delta n_{\uparrow} = -\Delta n_p$ here, different from the case where we considered the up- and down-spin electrons where we used $\Delta m = \Delta n_{\uparrow}$. The differing 1/2 factor comes from a simple consideration sketched on Fig. 3 where one can see that $\Delta N_{\uparrow,b} = 2\Delta N_{\uparrow,a}$.

This approach captures the intuitive physics at the commensurate-incommensurate phase transition: the spin susceptibility is diverging because of the Van Hove singularity in the density of states of the uncompensated up-spin electrons when total number of electrons is not fixed. However, the linear dispersion of bound pairs, which also holds at the critical field, quenches the above-mentioned divergence when the total number of electrons is kept fixed.¹⁴ Certainly the above simple model provides with the correct physical picture, but it is valid in dilute limit, so it does not capture the special role of the half filled band. For the weak coupling limit close to half filling we refer to Ref. 17.

VI. SUMMARY

We worked out effective field theory for the one-dimensional electron systems describing the vicinity of the magnetization plateau. First we treated free electrons at saturation magnetization and then we added interactions in the lowest order of coupling constant. We showed that away of half filling (for bipartite models that we considered) magnetization is a linear function of magnetic field at the edge of the saturation magnetization plateau if total number of electrons is not allowed to vary. The same theory describes also other than saturation plateaus, e.g., we described one encountered in dimerized doped Hubbard chain. Similar reasoning is easily generalizable to other plateaus supporting gapless nonmagnetic excitations in polymerized doped Hubbard chains.⁸ All these plateaus share similar feature: they are adiabatically connected with noninteracting cases, that is why they are relatively easy to treat. More subtle situation arises when plateaus are opened due to the interaction, thus one cannot describe them perturbatively. For such situations we explained in detail the connection between the zero magnetization plateau of attractively interacting two component Fermi systems in dilute limit¹⁷ and saturation magnetization plateau of free electron system.

The similar behavior of magnetization at the onset holds for spin 1 bosons,¹⁸ where the spin degrees of freedom for zero magnetization are described by the O(3) nonlinear sigma model with gapped spectrum,¹⁹ while charge degrees of freedom are gapless. Calculation simplifies in the dilute limit, which inherently falls under the strong coupling, where the system is made of strongly bound singlet pairs^{20,21} and the similar reasoning to that developed in Sec. V shows that magnetization increases linearly as soon as the magnetic field suppresses the spin gap if the total number of bosons is not allowed to vary. However to calculate for general situation magnetic susceptibility at the edge of zero magnetization plateau one should go beyond the spin-charge decoupling approximation (which follows from linearized hydrodynamics) in the effective formulation.

Very generally when gapless excitations are present at the plateau and they are not decoupled from the gapped magnetic excitations due to special microscopic symmetries,²² magnetization of isolated system shows linear growth with magnetic field instead of the square root behavior.

Our findings may be relevant for real quasi-one-dimensional spin gapped materials, where one can observe

experimentally crossover from square root dependence, at half filling, to linear growth of magnetization with magnetic field, away of half filling, in the vicinity of zero magnetization plateau.

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APPENDIX A: BOSONIZATION CONVENTIONS

Our bosonization rules are based on the following correspondence between the fermionic and bosonic operators:²³

$$\frac{c_{i,\sigma}}{a_0} \rightarrow \frac{\eta_\sigma e^{ik_{F,\sigma}x}}{\sqrt{2\pi}} e^{i\sqrt{4\pi}\Phi_{R,\sigma}(x)} + \frac{\eta_\sigma e^{-ik_{F,\sigma}x}}{\sqrt{2\pi}} e^{-i\sqrt{4\pi}\Phi_{L,\sigma}(x)}, \quad (\text{A1})$$

where $k_{F,\sigma} = \pi n_\sigma$ are Fermi momenta expressed in terms of fillings: $n_\sigma = \frac{1}{N} \langle \sum_{i=1}^N n_{i,\sigma} \rangle$, η_σ are Klein factors, ensuring fermionic commutation rules between electrons of different spin orientations,

$$\{\eta_\sigma, \eta_{\sigma'}\} = 2\delta_{\sigma,\sigma'}, \quad (\text{A2})$$

and $\Phi_{R,\sigma}$ and $\Phi_{L,\sigma}$ are right and left chiral bosonic fields of spin σ electrons with the following equal time commutation relations:

$$\begin{aligned} [\Phi_{R,\sigma}(x), \Phi_{L,\sigma'}(y)] &= \delta_{\sigma,\sigma'} \frac{i}{4}, \\ [\Phi_{R,\sigma}(x), \Phi_{R,\sigma'}(y)] &= \delta_{\sigma,\sigma'} \frac{i}{4} \text{sgn}(x-y), \\ [\Phi_{L,\sigma}(x), \Phi_{L,\sigma'}(y)] &= -\delta_{\sigma,\sigma'} \frac{i}{4} \text{sgn}(x-y). \end{aligned} \quad (\text{A3})$$

From these chiral fields we construct the usual bosonic fields and their dual counterparts,

$$\begin{aligned} \phi_\sigma(x) &= \Phi_{L,\sigma}(x) + \Phi_{R,\sigma}(x), \\ \theta_\sigma(x) &= \Phi_{L,\sigma}(x) - \Phi_{R,\sigma}(x), \end{aligned} \quad (\text{A4})$$

with nonlocal duality relation between them,

$$[\phi_\sigma(x), \theta_\sigma(y)] = i\Theta(y-x), \quad (\text{A5})$$

where $\Theta(x)$ is the Heaviside step function. Electron number operators are expressed in bosonization as follows:

$$n_{i,\sigma} \rightarrow n_\sigma + \frac{1}{\sqrt{\pi}} \partial_x \phi_\sigma(x) + \frac{1}{\pi} \sin[2k_{F,\sigma}x + \sqrt{4\pi}\phi_\sigma(x)]. \quad (\text{A6})$$

APPENDIX B: FREE ELECTRONS AT SATURATION MAGNETIZATION

Here we give a brief review of the magnetization process of free electrons in the vicinity of saturation magnetization.

For simplicity first we consider the grand canonical system, where the total number of electrons is not fixed. For external magnetic fields exceeding the saturation value, down-spin electron band gets completely empty in the ground state. With decreasing the magnetic field from its saturation value the magnetization will also decrease. Change in magnetization will have two contributions: (1) from the partially filled up-spin electrons and (2) from the dilute down-spin electrons. The contribution from partially field up-spin electrons is linear with the field and it is given by Eq. (6) (where for grand canonical systems one should omit the Lagrange multiplier). Let us discuss the contribution from the down-spin electrons. Close to the saturation the density of down-spin electrons is vanishingly small. In the absence of interactions the ground state energy density, given by its kinetic part, is

$$e_{0\downarrow} = \frac{E_{kin\downarrow}}{L} = \frac{\pi^2 n_{\downarrow}^3}{3}, \quad (\text{B1})$$

where $n_{\downarrow} = N_{\downarrow}/L = \Delta n_{\downarrow}$ and mass was normalized to 1/2 (we remind that $\hbar = 1$ throughout the paper). By definition chemical potential is given by the following expression: $\mu_{\downarrow} = \partial e_{0\downarrow} / \partial n_{\downarrow}$ which using Eq. (B1) gives

$$n_{\downarrow} = \frac{\sqrt{\mu_{\downarrow}}}{\pi} \quad (\text{B2})$$

and accordingly

$$\chi_{\downarrow}^{-1} = \left(\frac{\partial n_{\downarrow}}{\partial \mu_{\downarrow}} \right)^{-1} = 2\pi\sqrt{\mu_{\downarrow}}. \quad (\text{B3})$$

In particular we get that $\chi_{\downarrow}^{-1} \sim n_{\downarrow} \rightarrow 0$.

For free fermions $\mu(T=0) = E_F$, $N = L \int_0^{E_F} \chi d\mu$, thus χ is nothing else than density of states and its divergence at the bottom of the band is termed the Van Hove singularity in the density of states. Most importantly this divergence is a universal feature—this result is not restricted only to free fermions: interactions (short range, falling quicker than $1/x^2$) do not modify the above behavior. This one can check, e.g., from exact solution of XXZ spin chain. Indeed in XXZ model, close to saturation, magnetization depends as a square root on deviation of magnetic field from the saturation value and numerical prefactor in front of the square root is independent of the value of magnetic anisotropy (Z part of the exchange).

The relation between the particle density and chemical potential, given by Eq. (B2), holds also for massive relativistic fermions (Luther-Emery model), when chemical potential slightly exceeds the spectral gap, since the only ingredient needed for the above relation is the low energy dispersion, which must be quadratic in momenta [Eq. (21)]. The Luther-Emery model on the other hand is equivalent to the sine-Gordon model, which is a model of real scalar field self-interacting with $\cos \beta \phi$ term for particular value of $\beta = \sqrt{4\pi}$.^{11,12} This explains origin of Eq. (8) which follows from Eq. (B2), where for chemical potential one has to set $\Delta h/2 - \Delta\mu$, which couples to the density of down-spin electrons given in bosonization by $\partial_x \phi_{\downarrow} / \sqrt{\pi}$.

Now we discuss how the situation changes when the total number of electrons is kept fixed while changing the mag-

netic field across the saturation value. One can work out easily for free electrons that in order to keep the total number of electrons fixed while changing the magnetic field across the saturation value ($h = h_{sat} \rightarrow h_{sat} - \Delta h$) one has to adjust chemical potential (serving as Lagrange multiplier) accordingly,

$$\Delta\mu(h) = \frac{1}{2} \left[4 \left(1 - \cos \frac{\pi n}{2} \right) - \Delta h \right] + 2 \cos \left(\frac{\pi n}{2} + \arcsin \frac{4 \sin^2 \frac{\pi n}{2} - \Delta h}{4 \sin \frac{\pi n}{2}} \right). \quad (\text{B4})$$

One observes that for all fillings, except for half filling [note that at half filling $\Delta\mu(h) = 0$],

$$\left. \frac{\partial \mu(h)}{\partial h} \right|_{h=h_{sat}} = \frac{1}{2}. \quad (\text{B5})$$

Calculating the second derivative from Eq. (B4) we obtain

$$\left. \frac{\partial^2 \mu(h)}{\partial h^2} \right|_{h=h_{sat}} = -\frac{1}{2 \sin^2 \pi n}. \quad (\text{B6})$$

From Hamiltonian (3) we get for the down-spin electron density,

$$\langle \Delta n_{\downarrow} \rangle = -\Delta m = -\frac{1}{\pi} \sqrt{|\Delta\mu - \Delta h/2|} = -\frac{1}{\pi} \sqrt{\left| \frac{\partial \mu(h)}{\partial h} \Big|_{h_{sat}} \Delta h + \frac{1}{2} \frac{\partial^2 \mu(h)}{\partial h^2} \Big|_{h_{sat}} (\Delta h)^2 - \Delta h/2 \right|}. \quad (\text{B7})$$

Using Eqs. (B5) and (B6) we obtain

$$\langle \Delta n_{\downarrow} \rangle = -\frac{\Delta h}{2\pi \sin \pi n} = -\Delta m. \quad (\text{B8})$$

Thus recovering the anticipated result.

APPENDIX C: EXACT EXPRESSION FOR SATURATION SUSCEPTIBILITIES

Here for completeness we write out the Bethe ansatz results of the susceptibilities for repulsive^{15,22} and attractive^{13,14} Hubbard interactions at the saturation magnetization (with the correcting the misprint for attractive case). Close to the saturation magnetization ground state energy of the Hubbard model, being an analytic function, can be expanded in the Taylor series in infinitesimal deviation of the magnetization from its saturation value Δm ,

$$e_0\left(\frac{n}{2} - \Delta m\right) = e_{FM} - h_{sat}\Delta m + \frac{\chi_{sat}^{-1}}{2}(\Delta m)^2 + \dots, \quad (C1)$$

where saturation field is $h_{sat} = (\partial e_0 / \partial m)|_{m=n/2} = \lim_{\Delta m \rightarrow 0} ([e_{FM} - e(\frac{n}{2} - \Delta m)] / \Delta m)$ and the inverse saturation susceptibility is $\chi_{sat}^{-1} = (\partial^2 e_0 / \partial m^2)|_{m=n/2}$. From the above expansion it follows that in case $\chi^{-1} \neq 0$, $\Delta m \sim \Delta h$ and in case $\chi^{-1} = 0 \rightarrow \Delta m \sim \sqrt{\Delta h}$. The most general form allowed for the systems where ability to expand ground state energy into the Taylor series in magnetization can be proved is $\Delta m \sim (\Delta h)^{1/n}$, with n being positive integer.

Repeating the calculation presented in detail in Ref. 13 and correcting the missprint one obtains expressions of saturation susceptibilities,

$$\chi^{-1}(U < 0)|_n = \frac{2}{\pi} \sin(\pi n) \left[2\pi - 2 \arctan\left(\frac{4 \sin(\pi n)}{|U|}\right) \right]^2,$$

$$\chi^{-1}(U > 0)|_n = \frac{2}{\pi} \sin(\pi n) \left[2 \arctan\left(\frac{4 \sin(\pi n)}{U}\right) \right]^2. \quad (C2)$$

It is easy to verify that these expressions to first order of U indeed recover the bosonization result given by Eq. (19). We note as well that in the first order of U susceptibilities for both signs of U have identical form, in accordance with the bosonization, this way the typo was identified in the case of attractive interaction.^{13,14}

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